A SECOND EXAMPLE PROOF USING the WELL-ORDERING PRINCIPLE (Sec 5.3, #18)

<u>To Prove</u>: $5^n + 9 < 6^n$ for all integers $n \geq 2$.

<u>Proof</u>: Suppose, by way of contradiction, that there exists an integer $N \geq 2$ such that $5^N + 9 \geq 6^N$.

Let set S = { all integers t such that $t \geq 2$ and $5^t + 9 \geq 6^t$ }

By supposition, $N \geq 2$ and $5^N + 9 \geq 6^N$.

So, N is in Set S, and so, set S is not empty.

$\therefore$ Condition 1 of the Well-Ordering Principle is satisfied.

By definition of Set S, $t \geq 2$ for every integer t in S.

$\therefore$ Condition 2 of the Well-Ordering Principle is satisfied.

$\therefore$ By the Well-Ordering Principle, set S has a least element, m.

$\therefore$ $m \geq 2$ and $5^m + 9 \geq 6^m$ since m is in S.

Also, since m is the least integer in Set S, if integer $k < m$, then k is not in Set S.

[Proof Continues on the next page.]

[We show that $m \geq 3$ by showing that $m \neq 2$]

$5^2+9=34$, and $6^2=36$ and $34<36$.

$\therefore$ When $n=2$, $5^n+9 \leq 6^n$. That is, $5^2+9<6^2$.

But, since m is in Set S, $5^m+9 \geq 6^m$.

So, $m \neq 2$. $\therefore m \geq 3$

$\therefore m-1 \geq 2$

[We show that $5^{m-1}+9 < 6^{m-1}$]

Suppose, by way of contradiction, that $5^{m-1}+9 \geq 6^{m-1}$.

Since $m-1 \geq 2$ and $5^{m-1}+9 \geq 6^{m-1}$, $m-1$ is in Set S.

But $m-1<m$ and $m-1$ is in Set S, which contradicts the fact that m is the least integer element in Set S.

$\therefore 5^{m-1}+9 < 6^{m-1}$, by proof-by-contradiction.

$\therefore 5(5^{m-1}+9) < 5(6^{m-1}) < 6(6^{m-1}) = 6^m$

$\therefore 5^m+45 < 6^m$ [Subtract 36 from both sides]

$\therefore 5^m+9 < 6^m-36 < 6^m$

$\therefore 5^m+9<6^m$, but $5^m+9 \geq 6^m$, a contradiction.

[So, N never existed at the start!]

$\therefore$ For all integers $n \geq 2$,

$5^n+9<6^n$, by proof-by-contradiction.

QED